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4D EDGE CURRENTS FROM 5D CHERN-SIMONS THEORY

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ABSTRACT

A class of two dimensional conformal field theories is known to correspond to three dimensional Chern-Simons theory. Here we claim that there is an analogous class of four dimensional field theories corresponding to five dimensional Chern-Simons theory. The four dimensional theories give a coupling between a scalar field and an external divergenceless vector field and they may have some application in magnetohydrodynamics. Like in conformal theories they possess a diffeomorphism symmetry, which for us is along the direction of the vector field, and their generators are analogous to Virasoro generators. Our analysis of the abelian Chern-Simons system uses elementary canonical methods for the quantization of field theories defined on manifolds with boundaries. Edge states appear for these systems and they yield a four dimensional current algebra. We examine the quantization of these algebras in several special cases and claim that a renormalization of the $5D$ Chern-Simons coupling is necessary for removing divergences.

1 Introduction

The connection between three dimensional topological field theory and two dimensional conformal field theory is well known.[1] The latter is derivable from the former when the three dimensional domain for the topological theory has a boundary. Associated with this boundary are the so-called "edge states" which carry representations of a Kac-Moody algebra[2]. They have been shown to be relevant for the quantum Hall effect. [3, 4, 5]

Analogous edge states can also appear in higher dimensional topological field theory but not so much is known about these systems. Dynamics for higher dimensional topological field theory may be given by either a BF Lagrangian or a Chern-Simons Lagrangian. Edge states resulting from BF theory defined on a four dimensional space-time manifold with a boundary have been examined recently by members of the Syracuse group.[6] The relevance of these edge currents to the scaling limit of a superconductor was noted.

Here we analyze Chern-Simons theory defined on a five dimensional manifold with a boundary. After using canonical techniques developed in previous work along with Balachandran and Bimonte[5] and imposing suitable boundary conditions, edge states are seen to appear on the four dimensional space-time boundary. We shall argue that these edge states may be of some relevance in plasma physics.

Five dimensional Chern-Simons theory has been studied by a number of authors.[7, 8, 9] In particular, Floreanini, Percacci and Rajaraman in ref. [7] have examined Chern-Simons theories on five-manifolds $M = D \times R^1$, D being a four dimensional disc and R^1 accounting for time. More specifically D was taken to be a four dimensional solid ball and consequently ∂D was S^3. They then obtained a current algebra for the resulting edge states which was expressible in terms of field strengths F_{ij} evaluated at the boundary. A question remaining is

what is the role of these fields strengths. If they are to be considered as dynamical quantities we need to know their Poisson brackets (or more precisely, their Dirac brackets as second class constraints are present in the system.)

In section 2 we shall reproduce the current algebra of ref [7]. We find the added condition that the field strengths at ∂D should be considered as *nondynamical* or external fields. The condition results from demanding that the observable currents are everywhere differentiable. (Dropping such a requirement would lead to inconsistencies in the evaluation of the Poisson brackets.) Furthermore our results are general in that they apply to arbitrary four-discs D.

Because F_{ij} is fixed at ∂D, only one scalar degree of freedom survives at the boundary and it corresponds to $U(1)$ gauge transformations. We have found an effective four-dimensional field theory description of this system and it is analogous to a two dimensional conformal field theory. The effective Lagrangian yields the coupling of a scalar field $\phi(x)$ with the external vector field $B_k = \epsilon_{ijk}\; F_{ij}|_{\partial D}$. Since

$$\vec{\bigtriangledown} \cdot \vec{B} = 0 \;, \tag{1}$$

B_i might describe a magnetic field or the velocity vector field of an incompressible fluid.

The equation of motion following from the effective action is

$$\vec{B} \cdot \vec{\bigtriangledown} \partial_0 \phi = 0 \;, \tag{2}$$

∂_0 being a time derivative. It states that $\partial_0 \phi$ is constant along $B-$field lines. Such a situation may be encountered in magnetohydrodynamics. One possible example is that of an incompressible fluid constrained to flow in the same direction as an external magnetic field $\vec{B}(\vec{x})$. If we then set the velocity vector field $\vec{v}$ of the fluid equal to $\partial_0 \phi \vec{B}$, then the equation of motion (2) follows from (1) and the condition of incompressibility $\vec{\bigtriangledown} \cdot \vec{v} = 0$.

Another example of a system described by (1) and (2) occurs in the theory of rotating stars.[10] Here one starts with a noniuniformly rotating conducting fluid in the presence of a static magnetic field $\vec{B}(x)$. The field is assumed to be 'poloidal' which means that in cylindrical coordinates (r, θ, z) it has the form

$$\vec{B}(x) = (B_r(r,z), 0, B_z(r,z)\ ) \ , \tag{3}$$

subject to (1). The $z-$axis corresponds to the axis of rotation. The angular velocity $\omega(x)$ of the fluid is known to be governed by the isorotation law (due to Ferraro[11]), which is just Eq. (2) with the identification that $\partial_0 \phi = \omega$. This result is easy to show starting from the magnetostatic Maxwell equations and the relation for the electric current $\vec{J}$ in terms of the electromagnetic fields

$$\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \ , \tag{4}$$

σ being the conductivity. By taking the curl of (4) and using $\vec{\nabla} \times \vec{E} = 0$, one gets

$$\vec{\nabla} \times \vec{J} = \frac{1}{\sigma} \vec{\nabla}\sigma \times \vec{J} + \sigma \vec{\nabla} \times (\vec{v} \times \vec{B}) \ , \tag{5}$$

From the form of $\vec{B}(x)$ in (3) and using $\vec{\nabla} \times \vec{B} = 4\pi \vec{J}$, only the last term in (5) contributes in the polar (or $\theta-$) direction. Thus

$$\left(\vec{\nabla} \times (\vec{v} \times \vec{B}) \right)_\theta = 0 \ . \tag{6}$$

This equation is equivalent to (2) upon setting the $\theta-$component of $\vec{v}$ equal to $r\partial_0\phi$ and using (1).

Since the equation of motion (2) states that $\partial_0\phi$ is constant along $B-$field lines it follows that the solutions are unaffected by diffeomorphisms along the B direction. Furthermore, our effective action is invariant under such a restricted set of diffeomorphisms. These transformations are the analogue of two dimensional conformal transformations. We have found

the canonical expression for the generators of these diffeomorphisms in terms of the Chern-Simons fields.

In this article we also take up the quantization for five dimensional Chern-Simons theory in some simple cases (although the physical meaning of such an activity is not clear with regard to the above mentioned plasma physics applications). The resulting quantum mechanical commutation relations for the diffeomorphism generators are analogous to the Virasoro algebra.

To perform the quantization of this system we must specify two things: *i)* the topology of the boundary ∂D and *ii)* the external magnetic field $\vec{B}(\vec{x})$. After specifying *i)* we can write down an appropriate Fourier decomposition. This is done in Section 3. Here we shall look at two cases: A) ∂D equal to the $3-$torus T^3 (and hence D equal to a solid $3-$torus T^3) and B) $\partial D = S^2 \times S^1$.

With regards to *ii)* we initially let $\vec{B}(\vec{x})$ be arbitrary for case A) and obtain the classical current algebra and diffeomorphism algebra. For the latter a standard Sugawara construction can be employed to obtain the diffeomorphism generators. For case B) we take $\vec{B}(\vec{x})$ to be a magnetic monopole field. Our formalism applies for this system and allows for a quantization despite the use of an earlier assumption that two form constructed from F_{ij} is exact. We obtain the classical current algebra and diffeomorphism algebra also for this case. Here the analogue of the Sugawara construction for the diffeomorphism generators involves the $3-j$ symbols. We find that no quantization of the magnetic charge is required for a consistent quantization of this system.

The quantization is carried out in Section 4. Here we first look at the case of constant functions $\vec{B}(\vec{x})$ on T^3. We obtain a standard Fock space representation for the system. As

in conformal field theories, the quantum diffeomorphism algebra is seen have a central term. However here we find that the central term is divergent due to a sum over an infinite number of central charges. The result is analogous to defining the Virasoro algebra for strings in an infinite dimensional space-time. The divergence in the central term can be absorbed away by renormalizing the Chern-Simons coupling-but at the expense of loosing the noncentral term in the diffeomorphism algebra. The resulting algebra is thus a trivial one. We speculate that the same conclusion applies for any choice of *i)* and *ii)*.

In sec. 5 we give some concluding remarks concerning possible generalizations of this work.

2 The Canonical Formalism

We begin with the five-dimensional Chern-Simons action which is given by

$$S = \kappa \int_M A \wedge F \wedge F \tag{7}$$

where $F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = dA$ is the field strength two form. We choose the manifold M to be $D \times R^1$, with D being a four dimensional disc and R^1 corresponding to time. The Lagrangian density is

$$\mathcal{L} = \frac{\kappa}{4} \epsilon^{\mu\nu\lambda\rho\sigma} A_\mu F_{\nu\lambda} F_{\rho\sigma} \; . \tag{8}$$

It leads to both primary and secondary constraints in the Hamiltonian formalism. The former are

$$c^0 \;\; = \;\; \pi^0 \approx 0 \; , \tag{9}$$

$$c^i \;\; = \;\; \pi^i - \kappa \epsilon^{ijkl} F_{jk} A_l \approx 0 \; , \tag{10}$$

where π^μ are the canonical momenta, $i, j, .. = 1 - 4$ denote spatial indices and $\epsilon^{ijkl} = \epsilon^{0ijkl}$. Up to terms involving the constraints the Hamiltonian is given by

$$H = -3\kappa \int_D A_0 F \wedge F \ , \tag{11}$$

where we have set A_0 equal to zero on the boundary ∂D of D in order to eliminate the surface term. We can now compute the secondary constraints,

$$\begin{aligned} g^0 &= \{c^0, H\} = \frac{3\kappa}{4}\epsilon^{ijkl} F_{ij} F_{kl} \approx 0 \ , & (12) \\ g^i &= \{c^i, H\} = 3\kappa\epsilon^{ijkl}\partial_j A_0 F_{kl} \approx 0 \ , & (13) \end{aligned}$$

(12) is the analog of the Gauss law constraint in three dimensional Chern-Simons theory. From it we get that H is 'weakly' zero and that there are no further secondary constraints.

We next take up the constraint analysis.

2.1 Constraint Analysis

We can consistently set $A_0 = \pi^0 = 0$ in the above formalism, thereby eliminating the constraints (9) and (13). The remaining eight degrees of freedom A_i and π^i are then subject to five constraints. If D had no boundary the latter constraints could be expressed by (10) and (12). For D with a boundary, more care must be taken in defining the five remaining constraints. In particular, in order to compute their Poisson brackets we must insure that these constraints are differentiable.

To proceed we follow ref. [7] and introduce smearing functions $\Lambda = \Lambda(x)$ and $\Sigma = \Sigma(x)$ on D, where Λ is a scalar function and Σ is a one form, $\Sigma = \Sigma_i dx^i$. The constraints (10) and (12) are then replaced by

$$G(\Lambda) = \int_D d^4x \ \Lambda g^0 \ = \ 3\kappa \int_D \Lambda F \wedge F \approx 0 \ , \tag{14}$$

$$C(\Sigma) = \int_D d^4x\ \Sigma_i c^i \quad = \quad \int_D d^4x\ \Sigma_i \pi^i - 2\kappa \int_D \Sigma \wedge A \wedge F \approx 0\ . \tag{15}$$

The requirement that $G(\Lambda)$ and $C(\Sigma)$ be differentiable means that variations in A_i and π^i produce no surface terms. This is obviously the case for variations in π^i. By varying A in $G(\Lambda)$ and $C(\Sigma)$ we find

$$\delta G(\Lambda) \quad = \quad -6\kappa \int_D d\Lambda \wedge F \wedge \delta A + 6\kappa \int_{\partial D} \Lambda F \wedge \delta A\ , \tag{16}$$

$$\delta C(\Sigma) \quad = \quad 2\kappa \int_D (d\Sigma \wedge A - 2\Sigma \wedge F) \wedge \delta A - 2\kappa \int_{\partial D} \Sigma \wedge A \wedge \delta A\ . \tag{17}$$

One can now classify all possible boundary conditions for the fields and distributions consistent with the requirement that the surface terms in (16) and (17) vanish. We find however that only one set of such boundary conditions leads to a nontrivial current algebra. It is the following: For the distributions Λ and Σ we take

$$\Lambda|_{\partial D} = \Sigma|_{\partial D} = 0\ . \tag{18}$$

As in three dimensional Chern-Simons theory[5], imposing boundary conditions on the distribution functions is sufficient for the differentiability of the constraints. However, in order to have differentiable observables with a nontrivial current algebra we find it necessary to also impose boundary conditions on A_i. Imposing that the potentials vanish on the boundary is too strong as it then leads to trivial results. A weaker condition, which is the one we shall assume, is the requirement that the only dynamical degrees of freedom in A on the boundary ∂D correspond to $U(1)$ gauge degrees of freedom, or more generally shifts in A by a closed form on ∂D ,

$$A|_{\partial D} \rightarrow A|_{\partial D} + \omega \quad \text{where} \quad d\omega = 0\ . \tag{19}$$

(19) implies that the $U(1)$ curvature is fixed on the boundary.

We next obtain the algebra of the constraints. With the boundary terms in (16) and (17) vanishing, we get the following variational derivatives of $G(\Lambda)$ and $C(\Sigma)$:

$$\begin{aligned} \frac{\delta G(\Lambda)}{\delta A_l} &= -3\kappa\epsilon^{ijkl}\partial_i\Lambda F_{jk} \ , & \frac{\delta G(\Lambda)}{\delta\pi^i} = 0 \ , \qquad (20) \\ \frac{\delta C(\Sigma)}{\delta A_l} &= -2\kappa\epsilon^{ijkl}(\Sigma_i F_{jk} - \partial_i\Sigma_j A_k) \ , & \frac{\delta C(\Sigma)}{\delta\pi^i} = \Sigma_i \ . \qquad (21) \end{aligned}$$

Using these derivatives we obtain the following algebra:

$$\{G(\Lambda), G(\Lambda')\} = 0 \ , \qquad (22)$$

$$\{G(\Lambda), C(\Sigma)\} = -6\kappa\int_D d\Lambda\wedge\Sigma\wedge F \ , \qquad (23)$$

$$\{C(\Sigma), C(\Sigma')\} = -6\kappa\int_D \Sigma\wedge\Sigma'\wedge F \ , \qquad (24)$$

where we have assumed that all the smearing functions Λ, Σ, Λ' and Σ' vanish at ∂D.

From (22-24) the following linear combination of constraints is first class:

$$\begin{aligned} \mathcal{G}(\tilde\Lambda) &= C(d\tilde\Lambda) - G(\tilde\Lambda) \\ &= \int_D d^4x\ \partial_i\tilde\Lambda\pi^i - \kappa\int_D \tilde\Lambda F\wedge F \approx 0 \ . \qquad (25) \end{aligned}$$

Since $\tilde\Lambda$ in (25) appears as a smearing function for G, and since $d\tilde\Lambda$ appears as a smearing function for C, we need that

$$\tilde\Lambda|_{\partial D} = d\tilde\Lambda|_{\partial D} = 0 \ . \qquad (26)$$

The first class constraint (25) induces $U(1)$ gauge transformations which vanish on ∂D:

$$\begin{aligned} \delta_{\tilde\Lambda}A_i &= \{A_i, \mathcal{G}(\tilde\Lambda)\} = \partial_i\tilde\Lambda \ , \\ \delta_{\tilde\Lambda}\pi^i &= \{\pi^i, \mathcal{G}(\tilde\Lambda)\} = \kappa\epsilon^{ijkl}\partial_j\tilde\Lambda F_{kl} \ , \qquad (27) \end{aligned}$$

In the subsection which follows we obtain generators of $U(1)$ gauge transformations which are nontrivial on ∂D. These generators are in fact the observables for this system.

In addition to (25) are the first class constraints:

$$\mathcal{C}(\tilde{\Sigma}) = C(\tilde{\Sigma}_i) \approx 0 \ , \tag{28}$$

where here the distribution $\tilde{\Sigma}$ is not independent of the fields, but rather satisfies

$$\tilde{\Sigma} \wedge F = 0 \tag{29}$$

everywhere on D and $\tilde{\Sigma}$ vanishes on ∂D. Thus for such distributions we have $\mathcal{C}(\tilde{\Sigma}) = C(\tilde{\Sigma})$. Furthermore $\mathcal{C}(\tilde{\Sigma})$ will have nontrivial variations with respect to A_i as well as π^i,

$$\frac{\delta \mathcal{C}(\tilde{\Sigma})}{\delta A_l} = 2\kappa \epsilon^{ijkl} \partial_i \tilde{\Sigma}_j A_k \ , \quad \frac{\delta \mathcal{C}(\tilde{\Sigma})}{\delta \pi^i} = \tilde{\Sigma}_i \ . \tag{30}$$

The first class constraints (28) induce the following transformations on A_i and π^i:

$$\begin{aligned} \delta_{\tilde{\Sigma}} A_i &= \{A_i, \mathcal{C}(\tilde{\Sigma})\} = \tilde{\Sigma}_i \ , \\ \delta_{\tilde{\Sigma}} \pi^i &= \{\pi^i, \mathcal{C}(\tilde{\Sigma})\} = 2\kappa \epsilon^{ijkl} \partial_j \tilde{\Sigma}_k A_l \ . \end{aligned} \tag{31}$$

They too vanish on ∂D.

As noted in [7], the field strength matrix $[F_{ij}]$ satisfying the constraint $g^0 = 0$ is degenerate with a maximum rank of two. The same is true for the dual of F_{ij}. Therefore there are at least two independent solutions $\tilde{\Sigma} = \tilde{\Sigma}^{(1)}, \tilde{\Sigma}^{(2)}$ to (29), and there are at least two first class constraints of the form (28). A linear combination of these constraints along with the first class constraint (25) was shown to generate diffeomorphisms in the interior of D. It is given by

$$\Delta(w^{(0)}) = \mathcal{C}(i_{w^{(0)}} F) + \mathcal{G}(i_{w^{(0)}} A) \ , \tag{32}$$

where $w^{(0)}$ is a vector valued function which is required to vanish on ∂D and $i_{w^{(0)}}$ denotes contraction with $w^{(0)}$. The distribution $\tilde{\Sigma} = i_{w^{(0)}} F$ satisfies (29) due to

$$\epsilon^{ijk\ell}(i_{w^{(0)}} F \wedge F)_{jk\ell} = 3\epsilon^{ijk\ell} F_{mj} F_{k\ell} w^{(0)m} = \frac{3}{4}\epsilon^{mjk\ell} F_{mj} F_{k\ell} w^{(0)i} \;, \tag{33}$$

which is (weakly) zero.

2.2 Observables and Current Algebra

In the above, the phase space is spanned by eight variables A_i and π^i subject to five constraints. Since at least three of the five constraints are first class, none of the eight phase space variables survive as gauge invariant field degrees of freedom (or classical observables) in the interior of D. If D had no boundary the number of gauge invariant degrees of freedom for this system would be finite.

Are there any additional classical observables for D with a boundary? Here we are asking for quantities with the following properties. i) The quantities should be invariant with respect to transformations (27) and (31). Equivalently, they should have zero Poisson brackets with all first class constraints. ii) They should be differentiable with respect to all of the phase space variables. iii) Finally they should be trivial for the case of no boundary ∂D.

One set of quantities satisfying all of the above properties is:

$$q_1(f) = \int_D d^4x \; \partial_i f \pi^i \;, \tag{34}$$

where $f = f(x)$ is a scalar distribution function on D. It is easy to check that $q_1(f)$ is invariant under $U(1)$ gauge transformations (27). Furthermore, $q_1(f)$ has zero Poisson brackets with (28) and hence property i) is satisfied. With regards to ii) $q_1(f)$ is clearly

differentiable with respect to A_i and π^i. For this to be true no restrictions need to be imposed on f at the boundary ∂D, unlike the distribution functions Λ and Σ. Finally iii), if D has no boundary we can write $q_1(f)$ as a linear combination of constraints, and hence it is (weakly) zero. Furthermore we note that for D with a boundary, any two of observables $q_1(f)$ and $q_1(f')$ are equivalent if the distribution functions f and f' have the same values on ∂D. This is since

$$q_1(f) - q_1(f') = C(df - df') - \frac{2}{3}G(f - f') \approx 0 \ , \quad f|_{\partial D} = f'|_{\partial D} \ . \tag{35}$$

Thus the set of all $q_1(f)$'s with functions f coinciding on the boundary defines an equivalence class of observables.

Another quantity with properties i-iii) is

$$q_2(f) = \kappa \int_D (fF + 2df \wedge A) \wedge F \ , \tag{36}$$

where again we assume no restrictions on distribution function f at the boundary ∂D. With regards to i), $q_2(f)$ has zero Poisson brackets with all first class constraints. For ii) to be true we need to check variations with respect to A_i. We get

$$\delta q_2(f) = 2\kappa \int_D df \wedge F \wedge \delta A + 2\kappa \int_{\partial D} f d\delta A \wedge A \ . \tag{37}$$

The boundary term in (37) vanishes upon assuming that the variations of A are of the form (19). Hence $q_2(f)$ is differentiable, and this is the justification for the boundary condition (19). For the case of D with no boundary we can integrate (36) by parts to find $q_2(f) = -\frac{1}{3}G(f)$ which is (weakly) zero and thus iii) is satisfied. As before, $q_2(f)$ and $q_2(f')$ are equivalent if f and f' have the same values on ∂D, ie. $q_2(f) \approx q_2(f')$ for $f|_{\partial D} = f'|_{\partial D}$.

The next step is to compute the algebra of observables. Our approach is to look for Dirac variables. Thus we want quantities that in addition to satisfying i-iii), have the property iv)

that their Poisson brackets with the second class constraints are zero. Therefore they will have zero brackets with all constraints. Their current algebra can then be computed directly from their Poisson brackets.

We have found only one set of variables which satisfy i-iv). They are:

$$q(f) = q_1(f) + q_2(f) \ . \tag{38}$$

Thus

$$\{G(\Lambda), q(f)\} = \{C(\Sigma), q(f)\} = 0 \ , \tag{39}$$

for all Λ and Σ. In order to compute the Poisson bracket algebra of $q(f)$ we first need the variational derivatives,

$$\frac{\delta q(f)}{\delta A_\ell} = \kappa \epsilon^{ijk\ell} \partial_i f F_{jk} \ , \qquad \frac{\delta q(f)}{\delta \pi^i} = \partial_i f \ . \tag{40}$$

From them it follows that the $q(f)$'s are generators of $U(1)$ gauge transformations which, unlike those generated by $\mathcal{G}(\tilde{\Lambda})$, are nontrivial on ∂D. [For distributions f which vanish on ∂D, $q(f)$ is identical to the first class constraint $\mathcal{G}(f)$.] From (19) the degrees of freedom generated by $q(f)$ coincide with the degrees of freedom in $A|_{\partial D}$. The former are $U(1)$ gauge transformations connected to the identity.

Now from (40) we get the following current algebra

$$\begin{aligned} \{q(f), q(f')\} &= 4\kappa \int_D df \wedge df' \wedge F \\ &= 4\kappa \int_{\partial D} (f df' - f' df) \wedge F \ . \end{aligned} \tag{41}$$

We have thus recovered the same algebra as in [7]. The two form F appearing in (41) has no dynamics, since as stated earlier the $U(1)$ curvature is fixed on the boundary due to (19).

2.3 Effective Lagrangian

Since due to the boundary condition (19), $A|_{\partial D}$ contains only a scalar degree of freedom it makes sense that the above Chern-Simons system has an effective scalar field theory description. We shall now examine such a description.

The current algebra (41) which was gotten starting from the five dimensional Chern-Simons action is also obtainable from a Lagrangian written solely on the boundary ∂D. It is

$$L = 4\kappa \int_{\partial D} \partial_0 \phi \; d\phi \wedge F \; , \tag{42}$$

where ϕ is a scalar field. As before the curvature 2-form F is nondynamical on ∂D. In order to recover (41) one also needs to use that F is closed. (42) then represents the coupling of a scalar field to a static divergenceless vector field $B_i = \epsilon_{ijk} F_{jk}$. Physically B_i might describe a magnetic field or the velocity vector field of an incompressible fluid. The resulting equation of motion reads

$$\partial_0 \vec{\bigtriangledown} \cdot (\phi \vec{B}) = 0 \; , \tag{43}$$

or equivalently (2). General solutions for ϕ are of the form

$$\phi(\vec{x}, t) = \phi_0(\vec{x}) + \phi_1(\vec{x}, t) \quad \text{where} \quad \vec{B} \cdot \bigtriangledown \phi_1 = 0 \; . \tag{44}$$

ϕ_0 and ϕ_1 are analogous to the chiral modes of two dimensional conformal field theory.

We now show how to recover the current algebra (41) from the Lagrangian (42). In the Hamiltonian formalism, the momenta $\Pi(x)$ conjugate to $\phi(x)$ are constrained by

$$\Pi - 2\kappa \vec{B} \cdot \vec{\bigtriangledown} \phi \approx 0 \; , \tag{45}$$

and there are no further constraints. Let us assume that B_i is nonvanishing everywhere. Then (45) are second class and they are eliminated by finding Dirac variables. It is easy to

obtain such variables which we denote by $\rho(x)$,

$$\rho = \Pi + 2\kappa \vec{B} \cdot \vec{\nabla}\phi \ . \tag{46}$$

Finally to recover algebra (41) one can now define $q(f)$ according to

$$q(f) = \int_{\partial D} f(x)\rho(x) \ , \tag{47}$$

and compute its Poisson brackets.

2.4 Diffeomorphisms

Finally we take up the topic of diffeomorphism invariance in the five dimensional Chern-Simons theory or equivalently the four dimensional effective field theory. This invariance is the analogue of two dimensional conformal symmetry.

We begin with the effective Lagrangian (42). Since the curvature 2-form F is nondynamical on ∂D, (42) is not invariant under all possible diffeomorphisms on ∂D,

$$\delta\phi = \mathcal{L}_W \phi \ , \tag{48}$$

$\mathcal{L}_W$ denoting a Lie derivative. Rather, it is only invariant for those diffeomorphisms on ∂D which are along one direction - the B_i direction,

$$W_i = \epsilon(x) B_i \ , \tag{49}$$

$\epsilon(x)$ being an arbitrary function on ∂D.

These diffeomorphisms on ∂D can be extended to all of D and their generators can be expressed in terms of A and F. The generators are given by

$$\Delta(w) = \int_D d^4x \ (\mathcal{L}_w A)_i c^i - \int_D d^4x (i_w A) g^0 + 2\kappa \int_{\partial D} (i_w A) F \wedge A \ , \tag{50}$$

where we require

$$w_i|_{\partial D} = W_i = \epsilon(x) B_i \ . \tag{51}$$

As in the above, $B_i = \epsilon_{ijk} F_{jk}$ is a tangent vector in ∂D. From (51) it also follows that

$$i_w F|_{\partial D} = 0 \ . \tag{52}$$

Upon examining the expression (50) for $\Delta(w)$, we note that we cannot write the first two terms as linear combinations of the constraints $C(\Sigma)$ and $G(\Lambda)$ because the relevant distribution functions $\Sigma = \mathcal{L}_w A$ and $\Lambda = i_w A$ do not vanish on ∂D. Using $\mathcal{L}_w = di_w + i_w d$ we can however rewrite (50) in terms of the first class constraints $\mathcal{C}(\tilde{\Sigma})$, with $\tilde{\Sigma} = i_w F$,

$$\Delta(w) = \mathcal{C}(i_w F) + \int_D d^4x \ \partial_i(i_w A)\pi^i - \kappa \int_D (i_w A) F \wedge F \ . \tag{53}$$

As required for distributions of $\mathcal{C}(\tilde{\Sigma})$, $\tilde{\Sigma} = i_w F$ satisfies (29) [due to (33)] and it vanishes on ∂D [due to (51)]. When $w|_{\partial D} = 0$, the second and third term in (53) can be identified with the first class constraint $\mathcal{G}(i_w A)$ (since then $i_w A|_{\partial D} = 0$) and we thus recover the generators (32) of difeomorphisms in the interior of D. The boundary term in (50) is needed for differentiability. Its variation in A is

$$2\kappa \ \delta \int_{\partial D} (i_w A) F \wedge A = 4\kappa \int_{\partial D} (i_w A) F \wedge \delta A \ . \tag{54}$$

The remaining terms in (50) produce the following boundary terms upon varying A:

$$\begin{aligned} & -2\kappa \int_{\partial D} \Big(\mathcal{L}_w A \wedge A + 3(i_w A) F \Big) \wedge \delta A \\ = \ & -2\kappa \int_{\partial D} \Big(di_w A \wedge A + i_w F \wedge A + 3(i_w A) F \Big) \wedge \delta A \\ = \ & -2\kappa \int_{\partial D} \Big(i_w F \wedge A + 2(i_w A) F \Big) \wedge \delta A \ , \end{aligned} \tag{55}$$

where we have used the boundary condition (19) on A. These boundary terms are cancelled by (54) after using the boundary condition (52).

We can now compute the variational derivatives of $\Delta(w)$. We find:

$$\begin{aligned} \frac{\delta\Delta(w)}{\delta A_\ell} &= \kappa\epsilon^{ijk\ell}\Big(2\partial_i(w^m F_{mj})A_k + \partial_i(i_w A)F_{jk}\Big) - \frac{1}{3}w^\ell g^0 + \partial_i w^\ell c^i - \partial_i(w^i c^\ell) \ , \\ \frac{\delta\Delta(w)}{\delta\pi^i} &= (\mathcal{L}_w A)_i \ . \end{aligned} \tag{56}$$

We can use the variational derivatives to obtain the following Poisson brackets of $\Delta(w)$ with the constraints:

$$\begin{aligned} \{\Delta(w), G(\Lambda)\} &= G(i_w d\Lambda) \ , \\ \{\Delta(w), C(\Sigma)\} &= C(\mathcal{L}_w\Sigma) \ . \end{aligned} \tag{57}$$

Thus $\Delta(w)$ have (weakly) zero brackets with all the constraints. Vanishing Poisson brackets with the first class constraints implies that $\Delta(w)$ are invariant with respect to transformations (27) and (31), while vanishing Poisson brackets with the second class constraints implies that $\Delta(w)$ are Dirac variables, or equivalently, their Dirac brackets are equal to their Poisson brackets. Now using (40) and (56) we can compute the Poisson brackets of $\Delta(w)$ with the observables $q(f)$. We find

$$\begin{aligned} \{\Delta(w), q(f)\} &= \int_D d^4x\Big(\partial_i(i_w df)c^i - (i_w df)g^0\Big) + 4\kappa\int_{\partial D}(i_w A)F\wedge df \\ &= q(i_w df) = q(\mathcal{L}_w f) \ . \end{aligned} \tag{58}$$

For the Poisson brackets of $\Delta(w)$ with $\Delta(w')$ we find

$$\{\Delta(w), \Delta(w')\} = \int_D d^4x\Big((\mathcal{L}_{w'}A)_j(\partial_i w^j c^i - \partial_i(w^i c^j)) - (w \rightleftharpoons w')\Big)$$

$$
\begin{aligned}
& -\int_D d^4x([i_w,\mathcal{L}_{w'}]A)g^0 + \frac{1}{3}G(i_{w'}i_wF) \\
& +2\kappa\int_D\Big(i_{w'}F\wedge A\wedge d(i_wF) + d(i_wA)\wedge F\wedge d(i_{w'}A) - (w\rightleftharpoons w')\Big) \\
= \; & \int_D d^4x\Big(d[i_w,\mathcal{L}_{w'}]A + [i_w,\mathcal{L}_{w'}]F\Big)_i c^i - \int_D d^4x([i_w,\mathcal{L}_{w'}]A)g^0 \\
& +2\kappa\int_{\partial D}([i_w,\mathcal{L}_{w'}]A)F\wedge A \\
= \; & \int_D d^4x\,(\mathcal{L}_{\mathcal{L}_w w'}A)_i c^i - \int_D d^4x(i_{\mathcal{L}_w w'}A)g^0 + 2\kappa\int_{\partial D}(i_{\mathcal{L}_w w'}A)F\wedge A \\
= \; & \Delta(\mathcal{L}_w w') \qquad (59)
\end{aligned}
$$

where $w'_i|_{\partial D} = \epsilon'(x)B_i$ and we utilized the identity $i_{\mathcal{L}_w w'}A = [i_w,\mathcal{L}_{w'}]A$. The Poisson brackets (58) and (59) are identical in form to the analogous ones in three dimensional Chern-Simons theory [5].

3 Mode Expansion

The remainder of this article is devoted to the quantization of this system.

For quantization we must specify what is the boundary ∂D of D and also the static two-form F on ∂D. As a prelude we first perform mode expansion of the current algebra (41) as well as the diffeomorphism algebra (59). We shall give the analogue of the Sugawara construction for the diffeomorphism generators.

We examine two cases: A) $\partial D = T^3$ ($= S^1\times S^1\times S^1$) with $F|_{\partial D}$ initially arbitrary and B) $\partial D = S^2\times S^1$ with $F|_{\partial D}$ equal to a magnetic monopole field. In our above canonical treatment F was assumed to be exact, and this will also be the case for our first example.

However in the second example, $F|_{\partial D}$ is closed but not exact. We will show that our formalism can be consistently extended to this case.

3.1 Case A) $\partial D = T^3$ and $F|_{\partial D}$ arbitrary

For a basis of test functions on the three-torus we choose

$$\chi^{(\vec{N})}(\vec{\theta}) = \exp\{i\vec{N}\cdot\vec{\theta}\}\ , \quad \vec{N} = (N_1, N_2, N_3)\ , \quad \vec{\theta} = (\theta_1, \theta_2, \theta_3)\ , \tag{60}$$

with $N_i =$ integers and $0 \le \theta_i < 2\pi$. Then an arbitrary $F|_{\partial D}$ can be expanded according to

$$F|_{\partial D} = \sum_{\vec{N}} \epsilon^{ijk} b_i^{(\vec{N})} \chi^{(\vec{N})}(\vec{\theta})\ d\theta_j \wedge d\theta_k\ , \tag{61}$$

where $\vec{b}^{(\vec{N})} = (b_1^{(\vec{N})}, b_2^{(\vec{N})}, b_3^{(\vec{N})})$ are constant vectors. From the closedness condition on F they satisfy

$$\vec{N}\cdot\vec{b}^{(\vec{N})} = 0\ , \tag{62}$$

for all $\vec{N}$.

In considering the observables $q(f)$, let $f^{(\vec{N})}$ be a distribution function defined on all D with the boundary value given by

$$f^{(\vec{N})}|_{\partial D} = \chi^{(\vec{N})}(\vec{\theta})\ . \tag{63}$$

All $q(f^{(\vec{N})})$ for $f^{(\vec{N})}$ satisfying the boundary condition (63) are (weakly) equivalent. Thus $q_{\vec{N}} = q(f^{(\vec{N})})$ defines an equivalence class of observables. Using (41) and (61) we can compute the Poisson bracket algebra for the $q_{\vec{N}}$'s. We find:

$$\{q_{\vec{N}}\ ,\ q_{\vec{M}}\} = (4\pi)^3 i\kappa\ (\vec{M} - \vec{N})\cdot\vec{b}^{(-\vec{N}-\vec{M})}\ . \tag{64}$$

From (62) it then follows that $q_{\vec{N}=(0,0,0)}$ is a central charge for any $\vec{b}^{(\vec{N})}$. Whether or not more such central charges exist depends upon the values for $\vec{b}^{(\vec{N})}$.

For the case of the diffeomorphism generators $\Delta(w)$, let $\vec{w}^{(\vec{N})}$ be a vector-valued distribution function on D with a boundary value given by

$$w_i^{(\vec{N})}|_{\partial D} = \frac{1}{4}\chi^{(\vec{N})}(\vec{\theta})\ \epsilon_{ijk}F^{jk}|_{\partial D} = \sum_{\vec{M}} b_i^{(\vec{M}+\vec{N})}\chi^{(\vec{M})}(\vec{\theta})\ . \tag{65}$$

All $\Delta_{\vec{N}} = \Delta(w^{(\vec{N})})$ for $w^{(\vec{N})}$ satisfying the boundary condition (65) are (weakly) equivalent and define an equivalence class of diffeomorphism generators. From (58) and (59) their Poisson brackets with $q_{\vec{N}}$ and with themselves are given by

$$\{\Delta_{\vec{N}}\ ,\ q_{\vec{M}}\} \ = \ \sum_{\vec{P}} i\vec{M}\cdot\vec{b}^{(\vec{P})}q_{\vec{M}+\vec{N}+\vec{P}}\ , \tag{66}$$

$$\{\Delta_{\vec{N}}\ ,\ \Delta_{\vec{M}}\} \ = \ \sum_{\vec{P}} i(\vec{M}-\vec{N})\cdot\vec{b}^{(\vec{P})}\Delta_{\vec{M}+\vec{N}+\vec{P}}\ . \tag{67}$$

These relations are realized if we set $\Delta_{\vec{N}}$ equal to

$$L_{\vec{N}} = \frac{1}{4^4\pi^3\kappa}\sum_{\vec{M}} q_{\vec{M}}q_{\vec{N}-\vec{M}} \tag{68}$$

and then apply the Poisson brackets (64). This corresponds to the Sugawara construction of the Virasoro generators. As in three dimensional Chern-Simons theory[5] we expect that $\Delta_{\vec{N}} \approx L_{\vec{N}}$.

3.2 Case B) $\partial D = S^2\times S^1$ with $F|_{\partial D}$ equal to a magnetic monopole field

We let θ and ϕ be the polar and azimuthal angles parametrizing S^2 with ψ parametrizing S^1. In terms of these coordinates the magnetic monopole two form is given by

$$F|_{\partial D} = g\sin\theta d\theta\wedge d\phi\ , \tag{69}$$

g being a constant magnetic charge. We note that it is possible to smoothly extend the two form F to the interior of the four dimensional disc D without introducing any singularities.

For this let us introduce the radial coordinate r, $0 \leq r \leq 1$, with $r = 1$ defining the boundary ∂D. We define a function $\hat{g}(r)$ such that

$$\hat{g} \to 0 \ , \quad \frac{d\hat{g}}{dr} \to 0 \ , \quad \text{as} \ \ r \to 0 \ , \tag{70}$$

$$\hat{g} \to g \ , \quad \frac{d\hat{g}}{dr} \to 0 \ , \quad \text{as} \ \ r \to 1 \ . \tag{71}$$

Now we can take

$$F = \hat{g}(r) \sin\theta d\theta \wedge d\phi - \frac{d\hat{g}}{dr} \cos\theta dr \wedge d\phi \ , \tag{72}$$

for all of D. (72) is a closed two form which coincides with (69) at $r = 1$ and is nonsingular at $r = 0$.

For a basis of test functions on ∂D we choose

$$Y_\ell^m(\theta, \phi) e^{in\psi} \ , \tag{73}$$

where $\ell = 0, 1, 2...$, $m = -\ell, -\ell + 1, ...\ell$ and n are integers. $Y_\ell^m(\theta, \phi)$ are the standard spherical harmonics. We let $f^{(n,m,\ell)}$ be a distribution function defined on all D with the boundary value given by (73). $q_{n,\ell,m} = q(f^{(n,\ell,m)})$ defines an equivalence class of observables. Using (41) and (69) we can compute the Poisson bracket algebra for the $q_{n,\ell,m}$'s. We find:

$$\{q_{n,\ell,m} \ , \ q_{n',\ell',m'}\} = -16\pi i g\kappa \ (-1)^m n \ \delta_{n,-n'} \delta_{\ell,\ell'} \delta_{m,-m'} \ . \tag{74}$$

Here $q_{0,\ell,m}$ are central charges and they carry representations of the rotation group.

Concerning the diffeomorphism generators $\Delta(w)$, let $\vec{w}^{(n,\ell,m)}$ be a vector-valued distribution function on D with a boundary value given by

$$w_\theta^{(n,\ell,m)}|_{\partial D} = w_\phi^{(n,\ell,m)}|_{\partial D} = 0 \ , \quad w_\psi^{(n,\ell,m)}|_{\partial D} = Y_\ell^m(\theta, \phi) e^{in\psi} \ . \tag{75}$$

Then $\Delta_{n,\ell,m} = \Delta(w^{(n,\ell,m)})$ for $w^{(n,\ell,m)}$ satisfying the boundary condition (75) define an equivalence class of diffeomorphism generators. Their Poisson brackets are given by

$$\{\Delta_{n,\ell,m}\ ,\ q_{n',\ell',m'}\} \;=\; in' \sum_{\ell''=|\ell-\ell'|}^{|\ell+\ell'|} \sum_{m''=-\ell''}^{\ell''} \alpha^{\ell,\ell',\ell''}_{m,m',m''}\, q_{n+n',\ell'',m''}\ , \tag{76}$$

$$\{\Delta_{n,\ell,m}\ ,\ \Delta_{n',\ell',m'}\} \;=\; i(n'-n) \sum_{\ell''=|\ell-\ell'|}^{|\ell+\ell'|} \sum_{m''=-\ell''}^{\ell''} \alpha^{\ell,\ell',\ell''}_{m,m',m''}\, \Delta_{n+n',\ell'',m''}\ . \tag{77}$$

In obtaining the above expressions we used the composition relation for spherical harmonics,

$$Y^m_\ell(\theta,\phi)Y^{m'}_{\ell'}(\theta,\phi) = \sum_{\ell''=|\ell-\ell'|}^{|\ell+\ell'|} \sum_{m''=-\ell''}^{\ell''} \alpha^{\ell,\ell',\ell''}_{m,m',m''}\, Y^{m''}_{\ell''}(\theta,\phi)\ . \tag{78}$$

The α coefficients are given by

$$\alpha^{\ell,\ell',\ell''}_{m,m',m''} = (-1)^{m''}\sqrt{\frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{4\pi}} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell'' \\ m & m' & -m'' \end{pmatrix}\ , \tag{79}$$

$\begin{pmatrix} \ell & \ell' & \ell'' \\ m & m' & -m'' \end{pmatrix}$ being the $3-j$ symbols. The α's satisfy the symmetry properties:

$$\alpha^{\ell,\ell',\ell''}_{m,m',m''} \;=\; \alpha^{\ell',\ell,\ell''}_{m',m,m''}\ ,$$

$$\alpha^{\ell,\ell'',\ell'}_{m,-m'',-m'} \;=\; (-1)^{m'+m''}\alpha^{\ell,\ell',\ell''}_{m,m',m''}\ ,$$

$$\sum_{\ell',m'} \alpha^{\ell_1,\ell_2,\ell'}_{m_1,m_2,m'}\, \alpha^{\ell',\ell_3,\ell}_{m',m_3,m} \;=\; \sum_{\ell',m'} \alpha^{\ell_3,\ell_2,\ell'}_{m_3,m_2,m'}\, \alpha^{\ell',\ell_1,\ell}_{m',m_1,m}\ . \tag{80}$$

The last identity follows from the decomposition of three spherical harmonics

$$Y^{m_1}_{\ell_1}(\theta,\phi)Y^{m_2}_{\ell_2}(\theta,\phi)Y^{m_3}_{\ell_3}(\theta,\phi) = \sum_{\ell,m,\ell',m'} \alpha^{\ell_1,\ell_2,\ell'}_{m_1,m_2,m'}\, \alpha^{\ell',\ell_3,\ell}_{m',m_3,m}\, Y^m_\ell(\theta,\phi)\ . \tag{81}$$

Using the identities (80) along with the Poisson brackets (74), we can verify that (76) and (77) are realized for $\Delta_{n,\ell,m}$ equal to

$$L_{n,\ell,m} = \frac{1}{32\pi g\kappa} \sum_{\ell',\ell'',n'} \sum_{m'=-\ell'}^{\ell'} \sum_{m''=-\ell''}^{\ell''} (-1)^{m'}\, \alpha^{\ell,\ell',\ell''}_{m,-m',m''}\, q_{n',\ell',m'}\, q_{n-n',\ell'',m''}\ , \tag{82}$$

This gives the Sugawara construction for case B).

4 Remarks on Quantization

Here we make some remarks concerning the quantization for cases A) and B) of the previous section. For both cases we obtain standard Fock space representations. We compute the central term in the quantum diffeomorphism algebra for one particular example and find that it is divergent. The divergence is due to the existence of an infinite number of central charges. It can be absorbed away by renormalizing the Chern-Simons coupling-but at the expense of loosing the noncentral term in the diffeomorphism algebra. The resulting algebra is then a trivial one. Since an infinite number of central charges exist in all of the examples which we study we speculate that similar conclusions apply there as well.

We begin our discussion with case A). For case A) we need to know the expression for $F|_{\partial D}$. Of course the simplest choice we could make would be to take all of its components (with respect to the θ_i coordinates) equal to constants. This corresponds to zero windings around the torus. We first consider this choice. Then we examine case A) when the components of $F|_{\partial D}$ (with respect to the θ_i coordinates) are not all equal to constants, thus allowing for non-zero windings around the three-torus. As a general analysis appears to be complicated we shall assume that only one nontrivial mode contributes to the expansion (61) for $F|_{\partial D}$.

4.1 Case A) – Zero Windings

Constant $F_{ij}|_{\partial D}$ or zero windings means that we make the following choice for the vectors $\vec{b}^{(\vec{N})}$:

$$\vec{b}^{(\vec{N})} = \delta_{\vec{N},(0,0,0)} \vec{b}^{(0)} \ . \tag{83}$$

By $\delta_{\vec{N},\vec{M}}$ we mean $\delta_{N_1,M_1}\delta_{N_2,M_2}\delta_{N_3,M_3}$. The choice (83) identically satisfies (62). From (64) the only nonzero Poisson brackets for the observables are between $q_{\vec{N}}$ and $q_{-\vec{N}} = q^*_{\vec{N}}$, where $\vec{N}\cdot\vec{b}^{(0)} \neq 0$. The variables are:

$$\{q_{\vec{N}}\ ,\ q^*_{\vec{N}}\} = -2(4\pi)^3 i\kappa\ \vec{N}\cdot\vec{b}^{(0)}\ . \tag{84}$$

The observables $q_{\vec{N}}$, where $\vec{N}\cdot\vec{b}^{(0)} = 0$, behave as central charges. (For example if we set $b^{(0)}_2 = b^{(0)}_3 = 0$, the central charges are given by $q_{(0,N_2,N_3)}$ for all values of $N_2,\ N_3$.)

For the quantum theory we replace (84) by the commutation relations

$$[Q_{\vec{N}}\ ,\ Q^\dagger_{\vec{N}}] = 2(4\pi)^3\kappa\ \vec{N}\cdot\vec{b}^{(0)}\ , \tag{85}$$

where $Q_{\vec{N}}$ is the quantum operator associated with $q_{\vec{N}}$. Thus if $\kappa > 0\ \ (\kappa < 0)$, $Q_{\vec{N}}$ for $\vec{N}\cdot\vec{b}^{(0)}\ > 0$ ($\vec{N}\cdot\vec{b}^{(0)}\ < 0$) are annihilation operators (upto a normalization) and $Q^\dagger_{\vec{N}}$ creation operators. The "vacuum" $|0>$ can be defined as usual by

$$Q_{\vec{N}}\mid 0>= 0 \quad \text{if} \quad \kappa\ \vec{N}\cdot\vec{b}^{(0)}\ > 0\ . \tag{86}$$

The excitations are obtained by applying $Q^\dagger_{\vec{N}}$ to the vacuum.

The quantum diffeomorphism generators are normal ordered versions of their classical counterparts (68),

$$\mathbf{L}_{\vec{N}} = \frac{1}{4^4\pi^3\kappa}\sum_{\vec{M}} : Q_{\vec{M}}Q_{\vec{N}-\vec{M}} : \quad . \tag{87}$$

Let us now choose $\kappa > 0$. Then we have that $\mathbf{L}_{\vec{N}}|0>= 0$ for $\vec{N}\cdot\vec{b}^{(0)}\ > 0$. From (83) the $\mathbf{L}_{\vec{N}}$'s yield an algebra which includes central terms,

$$[\mathbf{L}_{\vec{N}},\mathbf{L}_{\vec{M}}] = (\vec{N}-\vec{M})\cdot\vec{b}^{(0)}\ \mathbf{L}_{\vec{N}+\vec{M}} + \gamma(\vec{N})\ \delta_{\vec{N}+\vec{M},(0,0,0)}\ . \tag{88}$$

$\gamma(\vec{N}) = -\gamma(-\vec{N})$ is a function of the three dimensional lattice. Constraints on $\gamma(\vec{N})$ come from the Jacobi identity

$$[[\mathbf{L}_{\vec{N}}, \mathbf{L}_{\vec{M}}], \mathbf{L}_{-\vec{N}-\vec{M}}] + [[\mathbf{L}_{\vec{M}}, \mathbf{L}_{-\vec{N}-\vec{M}}], \mathbf{L}_{\vec{N}}] + [[\mathbf{L}_{-\vec{N}-\vec{M}}, \mathbf{L}_{\vec{N}}], \mathbf{L}_{\vec{M}}] = 0 \ ,$$

which gives

$$(\vec{N} - \vec{M}) \cdot \vec{b}^{(0)} \ \gamma(\vec{N} + \vec{M}) = (2\vec{M} + \vec{N}) \cdot \vec{b}^{(0)} \ \gamma(\vec{N}) - (2\vec{N} + \vec{M}) \cdot \vec{b}^{(0)} \ \gamma(\vec{M}) \ . \tag{89}$$

One immediate consequence of this equation is that $\gamma(\vec{N}) = \gamma(\vec{M})$ if $\vec{N} \cdot \vec{b}^{(0)} = \vec{M} \cdot \vec{b}^{(0)} \neq 0$. Thus for $\vec{N} \cdot \vec{b}^{(0)} \neq 0$, $\gamma(\vec{N})$ can be a function only of $\vec{N} \cdot \vec{b}^{(0)}$.

For the case of $\vec{N} \cdot \vec{b}^{(0)} = 0$, the consistency condition (89) implies that

$$2\gamma(\vec{N}) = \gamma(\vec{M}) - \gamma(\vec{M} + \vec{N}) \ , \quad \text{for } \vec{M} \cdot \vec{b}^{(0)} \neq 0 \ .$$

But since $\vec{M} \cdot \vec{b}^{(0)} \neq 0$, $\gamma(\vec{M})$ is a function only of $\vec{M} \cdot \vec{b}^{(0)}$, and furthermore it is identical to $\gamma(\vec{M} + \vec{N})$. It therefore follows that

$$\gamma(\vec{N}) = 0 \quad \text{for } \vec{N} \cdot \vec{b}^{(0)} = 0 \ . \tag{90}$$

Returning to the case of $n = \vec{N} \cdot \vec{b}^{(0)} \neq 0$, (89) gives a recursion relation for generating all $\gamma(\vec{N}) = h(n)$, starting with the values of γ at different points on the three dimensional lattice. The recursion relation for $h(n)$ is identical to that for the central term in the Virasoro algebra defined for the one dimensional lattice[12]. Its solution is well known,

$$h(n) = c_1 n + c_3 n^3 \ . \tag{91}$$

[We note that for the one dimensional lattice n takes on integer values and that this is not in general true for our case. However, if we assume that all of the components of

$\vec{b}^{(0)}$ are rationally related, then our h can be redefined so that its arguments are integers and consequently the solution (91) applies. The form (91) for h is unaffected by such a redefinition.]

It remains to calculate the coefficients c_1 and c_3. For this it is sufficient to compute the matrix element $< 0|\mathbf{L}_{\vec{N}}\mathbf{L}_{-\vec{N}}|0 >$ for two different values of $n = \vec{N}\cdot\vec{b}^{(0)} > 0$. Let us first choose $\vec{N} = \vec{N}^{(1)}$ so that $\nu_1 = \vec{N}^{(1)}\cdot\vec{b}^{(0)}$ gives the smallest positive value for n. We get

$$< 0|\mathbf{L}_{\vec{N}^{(1)}}\mathbf{L}_{-\vec{N}^{(1)}}|0 >= 2\nu_1 < 0|\mathbf{L}_{(0,0,0)}|0 > ,$$

or

$$h(\nu_1) = 0 \; . \tag{92}$$

Then choose $\vec{N} = \vec{N}^{(2)}$ so that $\nu_2 = \vec{N}^{(2)}\cdot\vec{b}^{(0)}$ is the next smallest positive value for n, $\nu_2 > \nu_1 > 0$. Here we get two possible answers for $h(\nu_2)$ depending on whether or not the value of ν_2 is twice that of ν_1,

$$h(\nu_2) = \begin{cases} \frac{\mathcal{D}}{4}\nu_1(\nu_2-\nu_1) & \text{for} \;\; \nu_2 \neq 2\nu_1 \; , \\ \frac{\mathcal{D}}{2}(\nu_1)^2 & \text{for} \;\; \nu_2 = 2\nu_1 \;\; , \end{cases} \tag{93}$$

and thus from (92) and (93),

$$\gamma(\vec{N}) = h(n = \vec{N}\cdot\vec{b}^{(0)}) = \begin{cases} \frac{\mathcal{D}}{4}\frac{\nu_1 n[n^2-(\nu_1)^2]}{\nu_2[\nu_2+\nu_1]} & \text{for} \;\; \nu_2 \neq 2\nu_1 \; , \\ \frac{\mathcal{D}}{12}\frac{n[n^2-(\nu_1)^2]}{\nu_1} & \text{for} \;\; \nu_2 = 2\nu_1 \;\; . \end{cases} \tag{94}$$

We note that this answer includes the case of $n = \vec{N}\cdot\vec{b}^{(0)} = 0$ since (90) is then satisfied. $\mathcal{D}$ in (93) and (94) counts the number of lattice points, labeled by $\vec{N}^{(0)}$, for which $\nu_0 = \vec{N}^{(0)}\cdot\vec{b}^{(0)} = 0$. But since we have an infinite lattice (and the components of $\vec{b}^{(0)}$ are rationally related), $\mathcal{D}$ *is divergent*. For this result we only need to find one nonzero $\vec{N}^{(0)}$ normal to $\vec{b}^{(0)}$

since then $\vec{N}^{(0)}$ times any integer is also normal to $\vec{b}^{(0)}$. The divergence in the central term is analogous to defining the Virasoro algebra for strings in an infinite dimensional space-time.

To deal with the divergence one can renormalize the coupling constant κ in $\mathbf{L}_{\vec{N}}$ by making the replacement:

$$\kappa \to \kappa_R = \kappa\sqrt{\mathcal{D}} \ . \tag{95}$$

If we assume a finite limit for κ_R when $\mathcal{D}$ tends to infinity, then all that remains of the algebra for the $\mathbf{L}_{\vec{N}}$'s is just the central term:

$$[\mathbf{L}_{\vec{N}}, \mathbf{L}_{\vec{M}}] \to \frac{1}{\mathcal{D}}\gamma(\vec{N})\ \delta_{\vec{N}+\vec{M},(0,0,0)} \ . \tag{96}$$

Hence we are left with the trivial result that the renormalized diffeomorphism generators satisfy a harmonic oscillator type algebra.

4.2 Case A) – Nonzero Windings

Now we take up case A) when the components of $F|_{\partial D}$ (with respect to the θ_i coordinates) are not all equal to constants. For simplicity we assume that only one nontrivial mode−

$$\chi^{(\vec{N}^{(0)})}(\vec{\theta}) \quad \text{(and its complex conjugate)}$$

−contributes to the expansion (61) for $F|_{\partial D}$. (The complex conjugate is needed for $F|_{\partial D}$ to be real.) Non-zero windings means that $\vec{N}^{(0)} \neq (0,0,0)$. We now replace (83) by

$$\vec{b}^{(\vec{N})} = (\delta_{\vec{N},\vec{N}^{(0)}} + \delta_{\vec{N},-\vec{N}^{(0)}})\vec{b}^{(0)} \ , \quad \vec{N}^{(0)} \cdot \vec{b}^{(0)} = 0 \ . \tag{97}$$

For this choice we have that $F|_{\partial D}$ is both real and closed. From (64) the only nonzero Poisson brackets are

$$\{q_{\vec{N}}\ ,\ q^*_{\vec{N}\pm\vec{N}^{(0)}}\} = -(4\pi)^3 i\kappa\ (2\vec{N} \pm \vec{N}^{(0)}) \cdot \vec{b}^{(0)} \ . \tag{98}$$

Here we find that $q_{k\vec{N}^{(0)}}$, $kN_i^{(0)}$ being integers for all i, are central charges.

Additional central charges may also be present in the system and this depends on the values of $\vec{b}^{(0)}$ and $\vec{N}^{(0)}$. For example let us take

$$\vec{b}^{(0)} = (\alpha, -\alpha, 0) \ , \quad \vec{N}^{(0)} = (1, 1, 0) \ . \tag{99}$$

Now if we denote

$$\hat{q}_n^{N_2,N_3} = q_{n+N_2,N_2,N_3} \ ,$$

then all of $\hat{q}_0^{N_2,N_3}$ are central charges. The Poisson brackets for $\hat{q}_n^{N_2,N_3}$, $n \neq 0$, are given by

$$\{\hat{q}_n^{N_2,N_3}, \hat{q}_m^{M_2,M_3}\} = -2(4\pi)^3 i\kappa\alpha \ n\delta_{n,-m}\mathcal{S}_{N_2M_2}\delta_{N_3,-M_3} \ , \tag{100}$$

where

$$\mathcal{S}_{NM} = \delta_{N+M,1} + \delta_{M+N,-1} \ . \tag{101}$$

This algebra is similar to what was obtained for the quantum Hall effect.[5] Only here we have an infinite number of Landau levels labeled by N_2 and N_3. The result is that there are an infinite number of edge currents. Following [5] $\mathcal{S}_{MN}$ can be diagonalized by a real orthogonal matrix $\mathcal{M}$ (as $\mathcal{S}_{MN}$ is real symmetric) and it has real eigenvalues λ_ρ. Setting

$$\tilde{q}_n^{N_3}(\rho) = \mathcal{M}_{\rho N_2}\hat{q}_n^{N_2,N_3} \tag{102}$$

we have

$$\{\tilde{q}_n^{N_2}(\rho), \tilde{q}_m^{M_2}(\sigma)\} = -2(4\pi)^3 i\kappa\alpha \ n\lambda_\rho\delta_{n,-m}\delta_{\rho,\sigma}\delta_{N_3,-M_3} \ . \tag{103}$$

(103) can be readily quantized as in the previous example.

The quantum diffeomorphism generators are again normal ordered versions of their classical counterparts (68). Since as in the previous discussion we have an infinite number of central charges (eg., $q_{k\vec{N}^{(0)}}$) we again expect that a divergence will appear in the central term of the diffeomorphism algebra, which upon renormalization will lead to a trivial algebra.

4.3 Case B)

Finally we take up case B). Here the only nonzero Poisson brackets are between $q_{n,\ell,m}$ and $q_{-n,\ell,-m} = (-1)^m q^*_{n,\ell,m}$. [The complex conjugation property for the q's follows from the complex conjugation property for the spherical harmonics, $Y_\ell^{m*}(\theta,\phi) = (-1)^m Y_\ell^{-m}(\theta,\phi)$.] They are given by

$$\{q_{n,\ell,m}\ ,\ q^*_{n,\ell,m}\} = -16\pi i g\kappa\, n \quad . \tag{104}$$

The corresponding quantum mechanical commutation relations are

$$[Q_{n,\ell,m}\ ,\ Q^\dagger_{n,\ell,m}] = 16\pi g\kappa\, n \quad , \tag{105}$$

$Q_{n,\ell,m}$ being the operators corresponding to $q_{n,\ell,m}$. Now if $g\kappa > 0 \quad (g\kappa < 0)$, $Q_{n,\ell,m}$ for $n \ > 0$ ($n \ < 0$) are annihilation operators (upto a normalization) and $Q^\dagger_{n,\ell,m}$ creation operators. *We note that there is no quantization of the magnetic charge g for this system.*

The quantum diffeomorphism generators are now normal ordered versions of (82). As in the previous examples there are an infinite number of central charges ($Q_{0,\ell,m}$) and we thus expect that a divergence will appear in the central term of the diffeomorphism algebra, leading to a trivial algebra upon renormalization.

5 Concluding Remarks

Here we remark on possible generalizations of our work.

The first natural generalization is to go to the case of the nonabelian Chern-Simons theory in five dimensions. The constraint analysis for this case should proceed in an analogous fashion to that in Sec 2.1. A complication however arises in the imposition of boundary conditions on the fields. In the abelian theory we needed to require that the field strengths

are fixed at ∂D in order that the observables $q(f)$ be differentiable. In the nonabelian case however such a boundary condition has no gauge invariant meaning. It appears that there we must instead fix an entire orbit in the space of field strengths. Such orbits are generated by gauge transformations at ∂D and correspond to the classical degrees of freedom of the edge states. An effective field theory for these degrees of freedom should be a four dimensional analogue of the Wess-Zumino-Witten model. Furthermore the algebra of observables which one finds there should be analogous to the nonabelian Kac-Moody algebra.

Another possible extension of this paper involves admitting additional topological actions in five dimensions. For this we can introduce a *set* of potentials one-forms $\{A^{(1)}, A^{(2)}, A^{(3)}, ...A^{(n)}\}$ on the five-manifold M. We can then generalize the action (7) to

$$\kappa_{abc} \int_M A^{(a)} \wedge dA^{(b)} \wedge dA^{(c)} \ , \tag{106}$$

where κ_{abc} is symmetric with respect to all three indices. This is analogous to the Chern-Simons description of the quantum Hall effect where there a, b and c label the a^{th}, b^{th} and c^{th} Landau level.[5]

Rather than introducing more one forms on M we can define $p(>1)-$forms: $B^{(p)}$. The following topological actions are then possible:

$$\int_M A \wedge B^{(2)} \wedge B^{(2)} \ , \tag{107}$$

$$\int_M A \wedge F \wedge B^{(2)} \ , \tag{108}$$

$$\int_M A \wedge dB^{(3)} \ , \tag{109}$$

$$\int_M B^{(2)} \wedge B^{(3)} \ , \tag{110}$$

$$\int_M A \wedge B^{(4)} \ . \tag{111}$$

In general the action may be a linear combination of (7) and (107-111). With such a modification to the original dynamics there exists the possibility that the boundary conditions (19) on the field strengths may be relaxed and that further we may obtain more than just a single scalar degree of freedom on ∂D.

In a future work we plan to address some of the above modifications to the five dimensional Chern-Simons system.

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